American University of Beirut

MATH 201

Calculus and Analytic Geometry III

Fall 2008-2009

Final Exam - solution

Exercise 1 **a.** If $f(u,v,w)$ is a differentiable function and if $u = x-y$, $v = y-z$, and $w = z-x$, show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$; $\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$, and $\frac{\partial f}{\partial z} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$, and hence $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

b. Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y) = 3x - y + 6$ on the circle $x^2 + y^2 = 4$

$$\nabla f = \lambda \nabla g \Leftrightarrow \begin{cases} 3 = 2\lambda x \\ -1 = 2\lambda y \\ x^2 + y^2 = 4 \quad (g(x,y) = 0) \end{cases} \Leftrightarrow \begin{cases} \frac{3}{2\lambda} = x \\ -\frac{1}{2\lambda} = y \\ x^2 + y^2 = 4 \end{cases}$$

$x^2 + y^2 = 4 \Rightarrow \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 4 \Rightarrow \lambda^2 = \frac{10}{16} \Rightarrow \lambda = \pm\frac{\sqrt{10}}{4}$

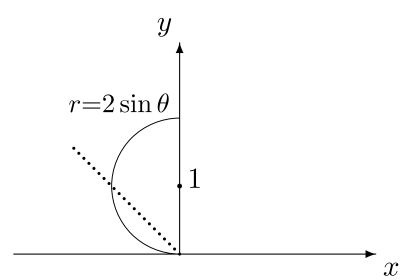
for $\lambda = \frac{\sqrt{10}}{4}$; $x = \frac{6}{\sqrt{10}}$, and $y = -\frac{2}{\sqrt{10}}$, and $f(x,y) = 2\sqrt{10} + 6$ (maximum value of f)

for $\lambda = -\frac{\sqrt{10}}{4}$; $x = -\frac{6}{\sqrt{10}}$, and $y = +\frac{2}{\sqrt{10}}$, and $f(x,y) = -2\sqrt{10} + 6$ (minimum value of f)

Exercise 2 Convert to polar coordinates, then evaluate the following integral

$$\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^0 xy^2 dxdy = \int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r^4 \cos\theta \sin^2\theta \; drd\theta$$

$$= \int_{\pi/2}^{\pi} \frac{32}{5} \cos\theta \sin^7\theta \; d\theta \; = \frac{4}{5}\left[\sin^8\theta\right]_{\pi/2}^{\pi} = -\frac{4}{5}$$

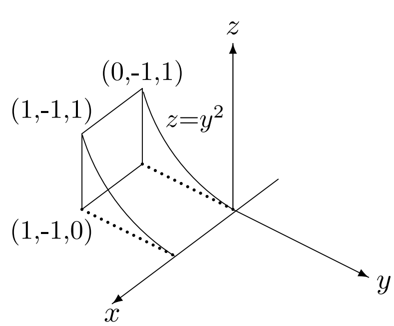


Exercise 3 $\int_0^1 \int_{-1}^0 \int_0^{y^2} dz \; dy \; dx = 1/3$

$$\int_{-1}^0 \int_0^1 \int_0^{y^2} dz \; dx \; dy \; = \int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy \; dz \; dx$$

$$= \int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy \; dx \; dz = \int_{-1}^0 \int_0^{y^2} \int_0^1 dx \; dz \; dy$$

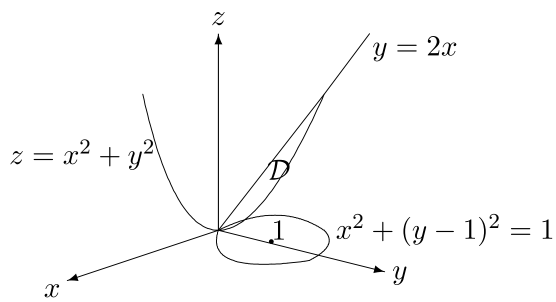
$$= \int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx \; dy \; dz$$



Exercise 4 Let V be the volume of the region D that is bounded by the paraboloid $z = x^2 + y^2$, and the plane $z = 2y$.

a) cartesian coordinates:

$$V = \int_0^2 \int_{-\sqrt{1-(y-1)^2}}^{\sqrt{1-(y-1)^2}} \int_{x^2+y^2}^{2y} dz\, dx\, dy$$



b) cylindrical coordinates:

$$V = \int_0^{\pi} \int_0^{2\sin\theta} \int_{r^2}^{2r\sin\theta} r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} (2r^2\sin\theta - r^3)\, dr\, d\theta = \int_0^{\pi} \left[\frac{2}{3}r^3\sin\theta - \frac{r^4}{4}\right]_{r=0}^{2\sin\theta} d\theta = \int_0^{\pi} \frac{4}{3}\sin^4\theta\, d\theta$$

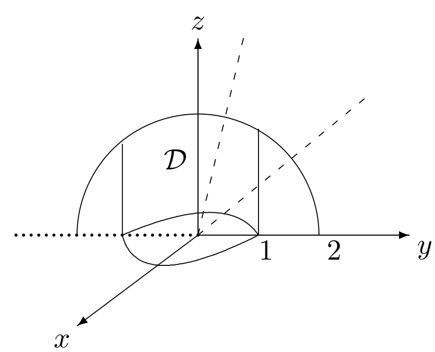
$$= \tfrac{4}{3}\cdot\tfrac{3\pi}{8} = \tfrac{\pi}{2}$$

Exercise 5 Let V be the volume of the region D that is bounded below by the xy-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$.

a) spherical coordinates: order $d\rho\, d\phi\, d\theta$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta\ +$$

$$\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\phi}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

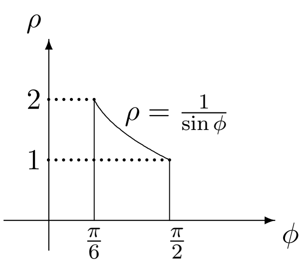


b) spherical coordinates: order $d\phi\, d\rho\, d\theta$

$$V = \int_0^{2\pi} \int_0^2 \int_0^{\frac{\pi}{6}} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta\ +$$

$$\int_0^{2\pi} \int_0^1 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta\ +$$

$$\int_0^{2\pi} \int_1^2 \int_{\frac{\pi}{6}}^{\sin^{-1}(1/\rho)} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta$$



the answer of part b) can also be written:

$$\int_0^{2\pi} \int_0^1 \int_0^{\frac{\pi}{2}} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta\ + \int_0^{2\pi} \int_1^2 \int_0^{\sin^{-1}(1/\rho)} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta \quad \text{(why??)}$$

Exercise 6 **a.** Find the work done by the force $F = x\mathbf{i} + y^2\mathbf{j} + (y - z)\mathbf{k}$ along the straight line from $(0,0,0)$ to $(1,1,1)$.

$r(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$; $\frac{dr}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $F(t) = t\mathbf{i} + t^2\mathbf{k}$, and $F \cdot \frac{dr}{dt} = t + t^2$, hence

$W = \int_0^1 (t + t^2)dt = 5/6$

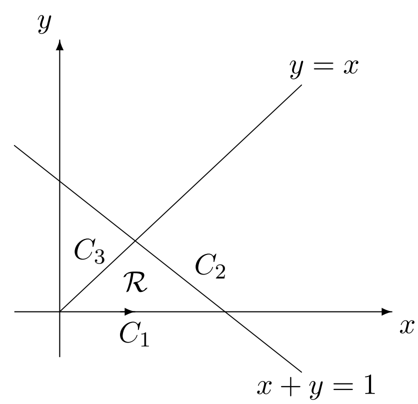
b. Evaluate

$$\int_{(0,0,1)}^{(1,\pi/2,e)} (\ln z + e^x \sin y)dx + e^x \cos y dy + (x/z - z)dz$$

$f(x,y,z) = x\ln z + e^x \sin y - \frac{z^2}{2} + C$ is a potential function (check it!), hence

$\int_{(0,0,1)}^{(1,\pi/2,e)} (\ln z + e^x \sin y)dx + e^x \cos y dy + (x/z - z)dz = \left[x\ln z + e^x \sin y - \frac{z^2}{2}\right]_{(0,0,1)}^{(1,\pi/2,e)} =$
$\frac{3}{2} + e - \frac{e^2}{2}$

c. Find the *outward flux* of the field $F = (y - 2x)\mathbf{i} + (x + y)\mathbf{j}$ across the curve C in the first quadrant, bounded by the lines $y = 0$, $y = x$ and $x + y = 1$.



i) direct calculation: $Flux = \oint_C Mdy - Ndx$

$C_1 : r_1(t) = t\mathbf{i}$, $0 \le t \le 1$; $Mdy - Ndx = -tdt$

and $\int_{C_1} Mdy - Ndx = \int_0^1 -tdt = -1/2$

$C_2 : r_2(t) = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1/2$;

$Mdy - Ndx = (1 - 3t)dt$

and $\int_{C_2} Mdy - Ndx = \int_0^{1/2} (1 - 3t)dt = -1/8$

$C_3 : r_3(t) = (\frac{1}{2} - t)\mathbf{i} + (\frac{1}{2} - t)\mathbf{j}$, $0 \le t \le 1/2$; $Mdy - Ndx = (\frac{3}{2} - 3t)dt$, and

$$\int_{C_3} Mdy - Ndx = \int_0^{1/2} (\frac{3}{2} - 3t)dt = 3/8$$

$Flux(F) = \oint_C Mdy - Ndx = -1/2 - 1/8 + 3/8 = -1/4$

ii) Green's theorem: $Flux = \int\int_R \mathbf{divF}dA$

$divF = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = -2 + 1 = -1$

$\int\int_R \mathbf{divF}dA = \int_0^{1/2}\int_y^{1-y} -\,dxdy = \int_0^{1/2} (2y - 1)dy = -1/4$